K-Essential Phantom Energy: Doomsday around the corner?

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Abstract

In spite of its rather weird properties which include violation of the dominant-energy condition, the requirement of superluminal sound speed and increasing vacuum-energy density, phantom energy has recently attracted a lot of scientific and popular interests. In this letter it is shown that in the framework of a general k-essence model, vacuum-phantom energy leads to a cosmological scenario having negative sound speed and a big-rip singularity, where the field potential also blows up, which might occur at an almost arbitrarily near time in the future that can still be comfortably accommodated within current observational constraints.

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The existence of phantom dark energy in the universe actually is a possibility not excluded by observations which has recently been widely discussed [1]. The physical properties of vacuum phantom energy are rather weird, as they include violation of the dominant-energy condition, $P + \rho < 0$,, naive superluminal sound speed and increasing vacuum-energy density. The latter property ultimately leads to the emergence of a singularity - usually referred to as *big rip* - in a finite time in future where both the scale factor and the vacuum-energy density blow up [2]. The existence of a singularity in finite time was already considered by Barrow, Galloway and Tipler in 1986 [3], even under the much weaker conditions $\rho > 0$ and $\rho + 3P > 0$, by assuming that $dP/d\rho$ is not a continuous function. This can actually be regarded as the first example of a big rip singularity. On the other hand, if we want the weak energy condition to be preserved one must regard the stuff of phantom energy to be made up of axions, at least when dealing with a quintessence field [4]. Indeed, if a quintessential scalar field ϕ with constant equation of state $P_\phi = \omega \rho_\phi$ is considered, then phantom energy can be introduced by allowing violation of dominant energy condition, $P_\phi + \rho_\phi < 0$, or what is equivalent, rotation of ϕ to imaginary values, $\phi \to i\Phi$, in the Lorentzian manifold [Notice that if the pressure and the energy density are respectively defined as $P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ and $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, with $V(\phi)$ the potential energy, then it follows that $P_\phi + \rho_\phi = (1 + \omega)\rho_\phi = \dot{\phi}^2$, and hence $\dot{\phi}^2 < 0$ (i.e. classically an axionic component for vacuum phantom energy [4]) if we want the weak energy condition to be satisfied also for $\omega < -1$, which will in turn automatically ensure violation of the dominant-energy condition]. No extra constraints are imposed to ensuring a causal propagation condition that $dP/d\rho$ does not exceed unity.

I will argue however that whereas $P_\phi + \rho_\phi < 0$ or $\phi \to i\Phi$ will suffice to ensure $\omega < -1$ and phantom energy, or *vice versa*, in all quintessence cases studied so far, the

eventual emergence of a big rip in the future will take only place in scalar-field models with equations of state of the simplest form $P_\phi = \omega \rho_\phi$. In fact, in case of a Chaplygin gas with equation of state $P_\phi = -A/\rho_\phi^n$ (where A and n are positive constants), which can in fact be regarded as just a particular case of a dust fluid with a given bulk viscosity in a k=0 Friedmann-Robertson-Walker universe [5], the existence of phantom energy does not lead to emergence of a big rip [6].

In this short letter I will show that a cosmological model with a singular big rip at an arbitrary finite time in the future can be also obtained when the scalar field satisfies equivalent phantom-energy conditions in the case that it is equipped with non-canonical kinetic energy for models restricted by a Lagrangian of the form

$$L = K(\phi)q(x),\tag{1}$$

where $x = \frac{1}{2}\Delta_\mu \phi \Delta_\nu \phi$. Such a definition, which of course includes the quintessence model as a limiting case, generally describes more general models claimed to solve the coincidence problem without fine tuning, which have been dubbed as k-essence [7]. Some of the current k-essence models featured suitable tracking behaviour during radiation domination with further attractors [7]. Introducing the usual variable $y = 1/\sqrt{x}$ and re-expressing $q(x)$ as $q[x(y)] \equiv g(y)/y$, from the perfect-fluid analogy, we have for the pressure and energy density of a generic k-essence scalar field ϕ [8]

$$P_\phi(y) = \frac{K(\phi)g(y)}{y}\tag{2}$$

$$\rho_\phi(y) = -K(\phi)g'(y),\tag{3}$$

where the prime means derivative with respect to y. Now, the equation of state parameter and the effective sound speed can be shown to be given by the $K(\phi)$-independent

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expressions

$$\omega_\phi(y) = -\frac{P_\phi}{\rho_\phi} = -\frac{g(y)}{yg'(y)} \tag{4}$$

$$c_{s\phi}^2(y) = \frac{P_\phi'}{\rho_\phi'} = \frac{g(y) - yg'(y)}{y^2 g''(y)}. \tag{5}$$

In general, k-essence models are defined by taking $K(\phi) = \phi^{-2} > 0$ [8]. Thus, for the weak energy condition to hold it follows from Eq. (3) that in these models the function $g(y)$ must be decreasing. Moreover, in these models it is currently assumed that $c_{s\phi}^2 > 0$ and hence Eq. (5) implies that $g''(y) > 0$, i.e. $g(y)$ should be a decreasing convex function [8].

We set next the general form of the function $g(y)$ when we consider a phantom-energy k-essence field; i.e. when we introduce the following two phantom-energy conditions: $K(\phi) < 0$ and

$$P_\phi(y) + \rho_\phi(y) \equiv 2K(\phi)xdq(x)/dx < 0,$$

which are just the conditions that would follow, both at once, whenever ϕ is made imaginary as in the quintessence models [4]. However, since the kinetic term is non-canonical in the k-essence scenario, the above two conditions should be defined by themselves, not as being derived from the general formalism of k-essence by Wick rotating the scalar field, for otherwise both the variable y and hence the function $g(y)$ would turn out to be no longer real. In what follows we shall therefore introduce the above two phantom-energy conditions while keeping y and $g(y)$ real.

Now, the first of these conditions and Eq. (3) amount to $g'(y) > 0$ in order for satisfying the weak energy condition $\rho_\phi > 0$, and then from $g'(y) > 0$ and the second phantom-energy condition, we deduce that $g(y) > yg'(y)$. Whence using $g'(y) > 0$, it also follows that $g(y) > 0$. Therefore the function $g(y)$ should be an increasing concave function, that is we must also set $g''(y) < 0$. We have then from Eq. (5) that the

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square of the speed of sound should necessarily be definite negative. However, even though an imaginary sound speed would at first sight mean catastrophic accelerated collapse of inhomogeneities, such a kind of instability could still be avoided at least at the subhorizon scale by taking into account the dependence of the sound speed on the wavelength characterizing the instabilities [9]. Finally, it is also a consequence from the above two phantom-energy conditions that $\omega_\phi(y) < -1$.

A simplest family of g-functions satisfying the above requirements is (see Fig. 1 (I))

$$g(y) = By^\beta, \tag{6}$$

with B and β being given constants such that $B > 0$ and $0 < \beta < 1$. Actually a more general function $g(y)$ can be written as a polynomials $g(y) = \sum_i B_i y^{\beta_i}$, where the first term is given by Eq. (6) and all other extra terms are characterized with powers $0 < \beta_i < 1$ as well and coefficients $B > B_1 > B_2 > ....$ It is moreover worth mentioning that the polynomial $g(y)$ cases seem to be linked to the eight asymptotes discussed by Barrow [10] when applying Fowler theorems for first-order differential equations to obtain solutions of the Raychaudhuri equation which are continuous, finite and monotonic as $t \rightarrow \infty$. Even though rigorously checking whether or not such a connection actually exist is outside the scope of the present work, it would appear interesting to investigate it. However, for the aims of this letter it will suffice taking only the first term of such a polynomials. Let us specialize then in the case of a spatially flat Friedmann-Robertson-Walker spacetime with line element

$$ds^2 = -dt^2 + a(t)^2 dr^2, \tag{7}$$

in which $a(t)$ is the scale factor. In the case of a universe dominated by a k-essence phantom vacuum energy, the Einstein field equations are then

$$3H^2 = \rho_\phi(y), \quad 2\dot{H} + \rho_\phi(y) + P_\phi(y) = 0, \tag{8}$$
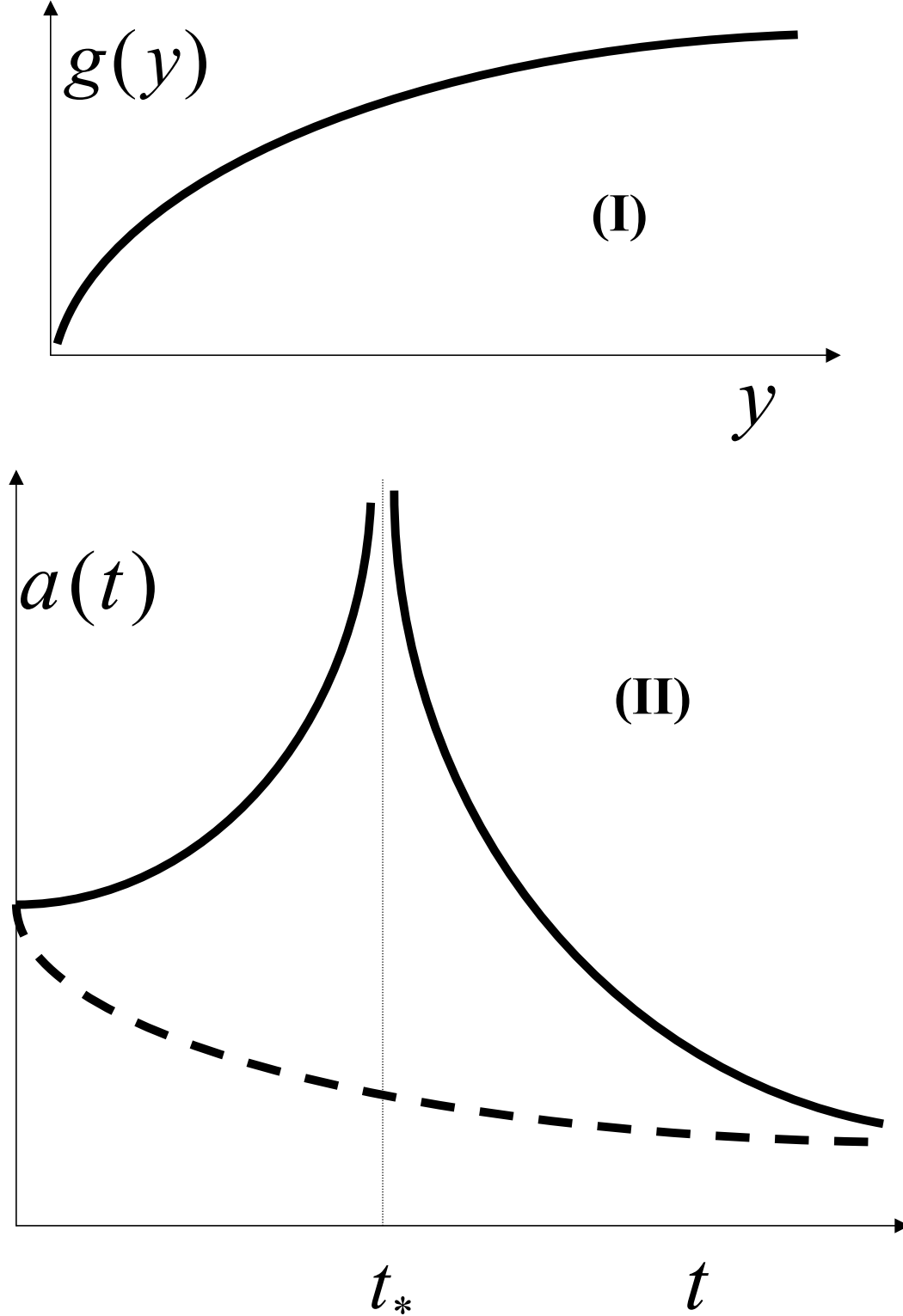
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Figure 1: (I) Generic shape of the function $g(y)$ for a k-essence vacuum phantom-energy field. All units in the plot are arbitrary. (II) Evolution of the scale factor $a(t)$ with cosmological time t for a function $g(y)$ with the form given in Fig. 1 (I). The dashed line for a universe with an ever decreasing size corresponds to a choice of the sign of the integration constant such that $t_* < 0$. If we choose that sign to be positive (solid line), then the constant t_* becomes an arbitrary time in the future at which the big rip takes place. All units in the plot of Fig. 1 (II) are also arbitrary.

with $H = \dot{a}/a$, the overhead dot meaning time derivative, $\dot{} = d/dt$. Combining the two expressions in Eq. (8) and using the equation of state we can obtain for the function $g(y)$ as given by Eq. (6)

$$3H^2 = \frac{2\dot{H}\beta}{1 - \beta}. \tag{9}$$

For our spatially flat case we have then the solutions

$$a \propto \frac{1}{(t - t_*)^{2\beta/[3(1-\beta)]}}, \quad 0 < \beta < 1, \tag{10}$$

where t_* is an arbitrary constant. If we choose $t_* < 0$, then the scale factor would ever decrease with time (see Fig 1 (II), dashed line). Obviously this solution family does not represent an accelerating universe and should therefore be discarded. Of quite greater interest is the choice $t_* > 0$ for which the universe (Fig. 1 (II), solid line) will first expand to reach a big-rip singularity at the arbitrary time $t = t_*$ in the future, to thereafter steadily collapse to zero at infinity; that is it matches the behaviour expected for current quintessence models with $\omega < -1$. The potentially dramatic difference is that whereas in quintessence models the time at which the big rip will occur depends nearly inversely on the absolute value of the state equation parameter, in the present k-essence model the time t_* is a rather arbitrary parameter.

In the case that we take for the field potential the usual expression $K(\phi) = \phi^{-2}$ [8], the Euler-Lagrange equation for the current k-essence field can also be written as

$$y^3 \frac{d^2 g(y)}{dy^2} \ddot{\phi} - 3Hy \left[y \frac{dg(y)}{dy} - g(y) \right] \dot{\phi} + \frac{4 \frac{dg(y)}{dy}}{\phi} = 0. \tag{11}$$

Therefore, using Eqs. (6) and (10) in the case that $K(\phi) = \phi^{-2}$ one can integrate Eq. (11) to obtain for the phantom-energy k-essence field

$$\phi = D_0 \left[a_0^{3/\beta} (t - t_*)^{\frac{\beta+1}{\beta-1}} + E_0 \right]^{\frac{\beta-1}{\beta+1}}, \tag{12}$$

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with D_0 and E_0 being arbitrary integration constants and a_0 an also arbitrary pre-factor for the scale factor in Eq. (10). We notice that the phantom field ϕ tends to vanish as $t \to t_*$, and hence its potential, $V = K(\phi) = \phi^{-2}$, blows up at the big rip, such as it happens in quintessence models [4]. It follows as well that the energy density for the phantom field will increase initially as $t \to t_*$ and blow up at $t = t_*$, as one should expect.

The main result in this letter is that phantom vacuum-energy leads to a big rip singularity also for k-essence dark energy. Moreover, one can play with the arbitrary values of the pre-factor a_0 for the scale factor expression in Eq. (10) and those un-boundedly small positive values of t_* which satisfy the observational constraint [11] $1 > \beta > 0.7$ (note that in the present model $\beta = -1/\omega_\phi$) and the currently observed cosmic acceleration rate [11] to check that such a set of present observations [11,12] is compatible with unboundedly small positive values of t_*. Even though unboundedly larger values of t_* are also allowed this way, k-essence phantom energy certainly may well allow a very near occurrence of the big rip in the future. Therefore, one could say that, in cosmological-time terms, a far and a near occurrence of the big rip are similarly probable and that, as a consequence from this, in the framework of cosmic k-essence, cosmic doomsday might be awaiting us around the corner.

References

1 R.R. Caldwell, Phys. Lett. B545 (2002) 23; A.E. Schulz and M.J. White, Phys. Rev. D64 (2001) 043514; J.G. Hao and X. Z. Li, Phys. Rev. D67 (2003) 107303; G.W. Gibbons, hep-th/0302199; S. Nojiri and S.D. Odintsov, Phys. Lett. B562 (2003) 147; B565 (2003) 1; B571 (2003) 1; P.Singh, M. Sami and N. Dadhich, Phys. Rev. D68 (2003) 023522; J.G. Hao and X.Z. Li, Phys. Rev. D68 (2003) 043501; 083514; M.P. Dabrowski, T. Stachowiak and M. Szydlowski, Phys. Rev. D68 (2003) 067301; E. Elizalde and J. Quiroga H., gr-qc/0310128; V.B. Johri, astro-ph/0311293

2 R.R. Caldwell, M. Kamionkowski and N.N. Weinberg, Phys. Rev. Lett. 91 (2003) 071301.

3 J.D. Barrow, G.J. Galloway and F.J. Tipler, Mon. Not. Roy. Astr. Soc. 223 (1986) 835 .

4 P.F. González-Díaz, *Axion Phantom Energy*, Report No. IMAFF-RCA-03-08; L.P. Chimento and R. Lazkoz, Phys. Rev. Lett. 91 (2003) 211301.

5 J.D. Barrow, Nucl. Phys. B310 (1988) 743 .

6 P.F. González-Díaz, Phys. Rev. D68 (2003) 021303(R); M. Bouhmadi and J.A. Jiménez-Madrid, in preparation.

7 C. Armendáriz-Picón, T. Damour and V. Mukhanov, Phys. Lett. B458 (1999) 209; J. Garriga and V. Mukhanov, Phys. Lett. B458 (1999) 219; T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D62 (2000) 023511; C. Amendáriz-Picón, V. Mukhanov and P.J. Steinhardt, Phys. Rev. Lett. 85 (2000) 4438; C. Amendáriz-

Picón, V. Mukhanov and P.J. Steinhardt, Phys. Rev. D63 (2001) 103510; L.P. Chimento and A. Feinstein, astro-ph/0305007

[**8**] . Malquarti, E.J. Copeland and A.R. Liddle, Phys. Rev. D68 (2003) 023512.

9 P.J. Steinhardt, *Quintessential Cosmology and Cosmic Acceleration*, http://feynman.princeton.edu

10 J.D. Barrow, Class. Quant. Grav. 13 (1996) 2965 .

11 D.N. Spergel *et al.*, Astrophys. J. Suppl. 148 (2003) 175.

12 S. Perlmutter *et al.*, Astrophys. J. 483 (1997) 565; S. Perlmutter *et al.*, Nature 391 (1998) 51; P.M. Garnavich *et al.* Astrophys. J. Lett. 493 (1998) L53; B.P. Schmidt, Astrophys. J. 507 (1998) 46; A.G. Riess *et al.* Astrophys. J. 116 (1998) 1009; A. Riess *et al.*, Astrophys. J. 560 (2001) 49; A.C. Baccigalupi, A. Balbi, S. Matarrase, F. Perrotta and N. Vittorio, Phys. Rev. D65 (2002) 063520; M. Melchiorri, L. Mersini, C.J. Odman and M. Tradden, Phys. Rev. D68 (2003) 043509; M. Doupis, A. Riazuelo, Y. Zolnierowski and A. Blanchard, Astron. Astrophys. 405 (2003) 409.